A Real Business Cycles Model with Changing Sentiments

by

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Abstract

In this paper the modification of the real business cycles model in which risk aversion parameter of agents' utility function follows bivariate markov chain is developed and estimated using simulated VAR. The model's ability to replicate properties of US quarterly data is compared with that of the standard real business cycles model. The main finding is that the model with markov switching performs at least well as the standard model. The model with markov switching also matches some features of the data which the standard RBC model is unable to match.

1. Introduction

The real business model, which is the basic model for the analysis of business cycles in a stochastic dynamic general equilibrium framework, assumes that the felicity (or instantaneous utility) function of the representative consumer does not change over time. Furthermore, most of the research works with a single-parameter class of such functions, namely either an isoelastic constant relative risk aversion function or its special case, the logarithmic function.

However, recent research in the field of asset pricing (Bakshi and Chen (1996) Campbell and Cochrane (1999a); Gordon and St-Armour (2000)) deals with models in which the felicity function does change over time. Specifically, those papers focus on time-varying attitudes towards risk in agents' preferences. This variation can be introduced either through dependence on some time-varying endogenous variable (wealth as in Bakshi and Chen (1996) or past consumption as in Campbell and Cochrane (1999a)) or through specification of an exogenous stochastic process which governs agents' attitude towards risk (as in Gordon and St-Armour (2000)). Variation introduced in this way into business cycle models might be thought of as allowing for agents to exhibit states of optimism and pessimism, and so one of the shocks driving the cycle can be described as due to changes in sentiment or confidence. A major finding of these papers is that allowing for a time-varying functional form of the instantaneous utility function improves the ability of the standard model (consumption CAPM in those cases) to explain asset prices behavior, at least in comparison with a specification where the instantaneous utility function does not change over time.

In this paper I develop a modification of the real business cycles model by introducing a time-varying instantaneous utility function. I adopt the Gordon and St-Armour (2000) approach to the specification of this variation. In order to follow the existing paradigm I model the utility function within the class of CRRA functions but allow the risk aversion parameter to change stochastically over time. Specifically, I assume that risk aversion follows a two-state Markov chain. It is worth noting here that changes in risk aversion in the general equilibrium context involving a utility function which depends on both consumption and leisure not only affects agents' attitude towards risk but also (and possibly to a greater extent) influences their labor – leisure choice.

The rest of the paper is organized as follows. In the next section I construct the model and derive equilibrium conditions. Section 3 demonstrates the technique I used to find the solution to the approximate log-linearized model, while section 4 describes the calibration and estimation procedures I used. Section 5 presents results of the estimation and section 6 concludes.

2. The model

The model is a stochastic dynamic general equilibrium model with perfect competition and market clearing in every period. It is similar to the real business cycle model with one major difference described below.

2.1 Consumers

The economy is populated by a large number of infinitely-lived homogeneous consumers with preferences over consumption of the final good *C* and labor services *L*, supplied to the market. These preferences can be represented by the following intertemporal utility function:

$$U_t = \sum_{k=0}^{\infty} b^k u_{t+k}(C_{t+k}, L_{t+k}) \tag{1}$$

The real business cycle theory assumes that the instantaneous utility function *u* does not change over time. The aim of this paper is to analyze an RBC model in which the parameters of the utility function change stochastically over time. Specifically, I consider a constant relative risk aversion utility function (CRRA) with the risk aversion parameter changing over time. These changes can be interpreted as changes to the level of the representative consumer's sentiment about prospects for the economy. The exact form of the utility function is the following:

$$u(C,L) = \frac{\mathrm{M}[(C\mathrm{M}^{-1}\exp(\frac{L^{1+s}}{r(1+s)}))^{1-r} - 1]}{1-r}, \qquad r > 1 \tag{2}$$

$$u(C,L) = \mathrm{M}[\log(C\mathrm{M}^{-1}) - \frac{L^{1+s}}{1+s}], \qquad r = 1 \tag{3}$$

where M is some positive constant and r is the relative risk aversion parameter. When r equals unity the utility function collapses to the logarithmic form. This utility function is a natural extension of the CRRA utility function with consumption as a single argument $u(C) = \frac{C^{1-r} - 1}{1-r}$. Labor supply is incorporated in exponential form inside the power function in order to preserve the constancy of the relative risk aversion $-\frac{U_{cc}(C,L)C}{U_c(C,L)}$, which is equal to r and at the same time ensure a constant labor supply elasticity with respect to real wage, which is equal to s^{-1}. The presence of constant M is specifically related to changes in r. The marginal utility of consumption is a multiple of M^r. Consequently, if r is constant over time then M^r also does not change and, therefore, M can be ignored. If, however r does change over time then the role of M is to scale changes in marginal utility of consumption which arise due to changes in r. In other words, since marginal utility of consumption can be represented as a function of the ratio C/M: $U_c(\frac{C}{M}, L) = \left(\frac{C}{M}\right)^{-r} \exp(\frac{L^{1+s}}{r(1+s)})^{1-r}$, by choosing the value of M one can control how large will the response of marginal utility of consumption be to changes of r for any given (for example, non-stochastic steady state equilibrium) values of consumption and labor supply.

The representative consumer's optimization problem can be divided into two parts reflecting choices at a point of time and across time. Firstly, he decides how much to work by equating the real wage W (taken by the consumer as given) to the ratio of the marginal disutility of work and the marginal utility of consumption:

$$-U_{L\,t}/U_{C\,t} = W_t \tag{4}$$

Secondly, at every period of time the consumer makes an intertemporal consumption-savings choice which is governed by the following Euler equation:

$$U_{C\,t} = b * \mathrm{E}_t[U_{C\,t+1}(1+r_{t+1})], \tag{5}$$

where r is the real interest rate which is taken by consumers as given.

2.2 The stochastic process for risk aversion

The risk aversion parameter follows a bivariate Markov chain with values for the two states of r^0 and r^1 and corresponding transition probabilities p_0 and p_1, i.e. for every t:

$$\Pr[r_{t+1}=r^0 | \, r_t=r^0]= p_0\,, \tag{6}$$

$$\Pr[r_{t+1}=r^1 | \, r_t=r^1]= p_1. \tag{7}$$

2.3 Firms

Firms operate in perfectly competitive markets with free entry and exit. Every firm has access to the following constant returns to scale Cobb-Douglas production technology:

$$Y_t=K_t^a(L_tZ_t)^{1-a}, \tag{8}$$

where K_t is a capital stock at the beginning of period t and Z_t is an exogenous stochastic technological process, the log of which follows an AR(1) process

$$\log(Z_t)=j\,\log(Z_{t-1})+e_t, \qquad e_t \text{ is n.i.d. } (0,\sigma^2) \tag{9}$$

Capital stock evolves according to the following dynamic capital accumulation equation:

$$K_{t+1}=(1-d)K_t+Y_t-C_t, \tag{10}$$

where d is the capital depreciation rate.

The firm's optimization problem and perfect competition results in the following first order conditions:

$$W_t=\P Y_t/\P L_t=(1-a)Z_t(K_t/Z_tL_t)^a, \tag{11}$$

$$r_t+d=\P Y_t/\P K_t=a(Z_tL_t/K_t)^{1-a}, \tag{12}$$

i.e. the marginal product of each factor is equated to its rental price.

2.4 Equilibrium conditions

Putting together equations (2)-(5), (8), (10)-(12) and eliminating factor prices one arrives at the following system which, together with the specification of a Markov switching process for the risk aversion parameter (6)-(7) and an autoregressive process for the technology (9), fully describes the dynamics of the economy:

$$K_{t+1}=(1-d)K_t+K_t^{a}(Z_tL_t)^{1-a}-C_t, \tag{13}$$

$$C_tL_t^{s}=(1-a)r_t(Z_tL_t/K_t)^{1-a} \tag{14}$$

$$(C_t\mathrm{M}^{-1})^{-r_t}\exp[\frac{L_t^{1+s}}{1+s}(1-\frac{1}{r_t})]=bE_t[(C_{t+1}\mathrm{M}^{-1})^{-r_{t+1}}\exp[\frac{L_{t+1}^{1+s}}{1+s}(1-\frac{1}{r_{t+1}})](1-d+a\frac{Z_{t+1}L_{t+1}}{K_{t+1}}^{1-a})] \tag{15}$$

Equation (13) is the capital accumulation constraint, (14) is the labor market equilibrium condition and (15) is the Euler equation which ensures intertemporal optimization.

3. Solution of the model

3.1 Non-stochastic steady state

The non-stochastic steady state is found by setting exogenous stochastic variables (the level of technology Z and the risk aversion coefficient ρ) to their unconditional means:

$$Z_t=Z^*=1,\ r_t=r^*=r^0\frac{1-p_1}{2-p_1-p_0}+r^1\frac{1-p_0}{2-p_1-p_0}\forall t$$

and finding the autonomous solution {K*,L*,C*} to the system (13)-(15). I follow the conventional notation that asterisks mean steady-state value. Since I am free to choose units of measurement, it is not the steady-state levels of three variables, but rather only two steady-state ratios, which are important for the subsequent analysis; these are the steady-state output-capital and consumption-output ratios. Simple algebra leads to the following:

$Y^*/K^*=(q+d)/a$, $\quad C^*/Y^*=1-ad/(q+d)$,

where $q=1/b-1$ is the subjective discount rate.

3.2 Solution to the linearized stochastic system

Because the model is highly nonlinear it is impossible to find its closed-form solution in the presence of uncertainty. The conventional way to solve a model of this kind is to log-linearize it around the non-stochastic steady state and to get the approximate system of linear first order difference equations. Under regularity conditions this system will exhibit the saddle path property, which allows one to express the control variable (consumption) as a function of the current period state variables (capital stock and technology parameter in the real business cycle model). Applying this method to the model presented above will lead to the representation of consumption as a function of the current period capital stock, technology and the risk aversion coefficient. However, the model considered here has one non-continuous variable – the risk aversion coefficient. Therefore, the actual realization of its value is never near its steady state value. Hence, I used a slightly different linearization technique to solve the model. Namely, I log-linearized the Euler equation (15) around the steady state levels of consumption, capital stock and labor supply, treating the risk aversion parameters r_t and r_{t+1} as constants. This yields the following:

$$\begin{aligned} &-r_t c_t + d_L\left(1-\frac{1}{r_t}\right)l_t = \\ &E_t\left[-r_{t+1}c_{t+1} + d_L\left(1-\frac{1}{r_{t+1}}\right)l_{t+1} + \frac{q+d}{1+q}(1-a)(l_{t+1}+z_{t+1}-k_{t+1}) - (r_{t+1}-r_t)m\right], \end{aligned} \tag{16}$$

where lower case variables indicate the log of deviations from the steady state value of the corresponding upper case variable, and

$$d_L = \frac{1-a}{C^*/Y^*} r^*, \qquad m = \log \frac{C^*}{\mathrm{M}}.$$

Linearization of the capital accumulation equation (13) and labor market equilibrium equation (14) is straightforward and gives the following:

$$k_{t+1} = (1-d + \frac{Y^*}{K^*} a)k_t + (1-a)\frac{Y^*}{K^*}(l_t + z_t) - (\frac{Y^*}{K^*} - d)c_t, \tag{17}$$

$$c_t + sl_t = a(k_t - l_t) + (1-a)z_t + \log \frac{r_t}{r^*}. \tag{18}$$

The process for technology in (9) is already in logs and transforms to the following:

$$z_{t+1} = j z_t + e_{t+1}. \tag{19}$$

Applying the method of undetermined coefficients to equations (16) – (19), together with the Markov chain law for risk aversion (6) – (7), one can get the following representation for consumption:

$$c_t = a_1 k_t + a_2 z_t + a_3 \text{ if } r_t = r^0 \tag{20}$$

$$c_t = a_4 k_t + a_5 z_t + a_6 \text{ if } r_t = r^1 \tag{21}$$

The linear form of solution is implied by the linear structure of the system (16) – (19). This solution differs from the simple log-linearization in the following way. In the notation of (20) – (21) the latter assumes that $a_1 = a_4$ and $a_2 = a_5$ so that only the intercept changes with the different realizations of the risk aversion parameter. The solution method of undetermined coefficients allows all coefficients to be different. The derivation of coefficients a_1- a_6 is presented in the appendix.

4. Calibration and estimation method

There are 11 unknown parameters in the model. Seven of them characterize consumers' preferences: the discount factor b, the utility function constant m, two possible values of the risk aversion coefficient r^0 and r^1, two transitional probabilities p_0 and p_1, and the inverse of the labor supply elasticity s. Another four unknown parameters characterize production technology: the capital share a, the capital depreciation rate d, the autoregressive coefficient of the stochastic technology process j, and the standard deviation of the shock in this process s^2. To assess the behavior of the model I used a mixed strategy. I calibrated five parameters and estimated the remaining six. For the calibration and estimation purposes I used quarterly US data for 1959-1999 on real per capita private consumption and real per capita private investment. I used the sum of these two as a measure of real output. The calibration was performed in several steps. Firstly, I set the subjective discount rate q to 3% per annum – the value, which is considered standard in the business cycle literature and is consistent with post-war data on the US short-term risk-free real interest rate. Secondly, taking into account the steady-state properties of the model, one can express a and d as functions of the steady-state values of the discount rate, the output – capital and the consumption – output ratios. I set these ratios to their average values in the actual data and thereby find estimates for a and d. Finally, for computational purposes I also choose to set the autoregressive coefficient j to a value of 0.92 and the inverse of the labor supply elasticity s to 0, which is consistent with the existing business cycle literature. Table 1 summarizes the calibrated parameters.

I estimate the remaining six parameters following the method described in Smith (1993). Briefly, this method suggests that the parameters of the model can be estimated by first choosing an auxiliary model and then matching up the parameters of the auxiliary model implied by the underlying model with the estimates of the auxiliary model parameters from the data. Specifically, he chooses a VAR as the auxiliary model and minimizes the weighted sum of squared differences between the VAR coefficients from the actual and simulated data; the weights used equal the inverse of the variances of the VAR coefficients estimated from the actual data. Thus, the minimization criterion is equal to

$$\sum_i (x_i^s - x_i^d)^2 \frac{1}{s_{x_i}^2},$$

where x is the vector of the VAR estimates (both coefficients and the variance-covariance matrix of errors), superscript s means estimates from simulated data, superscript d means estimates found from actual data, and s_x^2 is the vector of estimates of the variances of x obtained from fitting a VAR to actual data.

Because in the simulated data consumption, investment and output are always linearly dependent I choose to estimate a VAR on output and investment alone. The order of the VAR was set to one and the length of the simulated data was set to 5000 i.e. the parameters of the VAR corresponding to the model were found by fitting it to 5000 observations simulated from the underlying model.

5. Results

Table 2 presents results of the estimation of the remaining 6 parameters for two different models: the model described above and the model without Markov switching, i.e. a standard real business cycle model with a constant risk aversion parameter.

Analysis of Table 2 allows me to make several judgements about the estimates. For both models the estimated values of the risk aversion coefficient lie well within the range [1; 3] considered standard in the business cycle literature. At the same time, it is worth noting that the behavior of the model without Markov switching does not depend greatly on the value of r (i.e. the dynamic properties of the model do not differ much for different values of r within the range [1;3] holding other parameters constant). This is not the case for the Markov switching where even a small change in one of the possible states that r can take (holding the other constant) has an effect on the dynamics of the model. This influence comes through an impact on labor supply decisions and thus on the variance of the labor supply and all other variables. Both of the transition probabilities p_0 and p_1 are high and close to estimates obtained by Gordon and St – Armour (2000) in their similar partial equilibrium study of asset prices with a Markov switching utility coefficient. The estimate of s, the standard deviation of the technological shock, is lower in the Markov switching model than in the traditional RBC model. This is because there is an additional source of volatility in this model through the random change in the risk aversion coefficient. Therefore, to match the given stochastic properties of the data a less volatile technology shock would be enough. This is a desirable characteristic as the value of s in the standard RBC model has been criticized as implying a high probability of technical regress. The above result suggests that some of the volatility of output is indeed due to swings in sentiment.

The relative performance of the two alternative models can be assessed in several ways. First, one can compare the parameters of a VAR estimated from the simulated data for each of the two models with VAR estimates from the actual data. Table 3 presents the results of estimation of a VAR(1) for the actual and simulated data on output and investment. The matrix S (with elements s_{ij}) is the matrix of standard errors of estimates of the matrix A (with elements a_{ij}):

$$\begin{bmatrix} y_{t+1} \\ i_{t+1} \end{bmatrix} = A \begin{bmatrix} y_t \\ i_t \end{bmatrix} + w_{t+1},\ w_t \sim N(0, \Omega).$$

In order to assess the relative ability of the two models to match the VAR properties of the data I performed two Wald tests for the hypothesis that estimates of the elements of the matrix A from the US data are equal to those from the two models. The relevant Wald statistic, W, is equal to

$$W = vec[(\hat{A}^d)' - (\hat{A}^s)']' \left([\mathrm{cov}(\hat{\Omega}^d)]^{-1} \otimes \begin{bmatrix} \Sigma_1^{T-1}(y_t^d)^2 & \Sigma_1^{T-1}(y_t^d i_t^d) \\ \Sigma_1^{T-1}(y_t^d i_t^d) & \Sigma_1^{T-1}(i_t^d)^2 \end{bmatrix} \right) vec[(\hat{A}^d)' - (\hat{A}^s)'],$$

where superscript d stands for the actual data, superscript s – for the data simulated by the model, *vec* is the matrix operator which has a nxm matrix as an argument and produces a nmx1 vector which is given by stacking columns of the argument matrix on top of each other, and symbol $\otimes$ denotes the Kroneker product of two matrices. Under the null hypothesis that the true values of the elements of the matrix A are equal to their estimates from the VAR on the data simulated by the model the Wald statistics W is asymptotically distributed as $c^2(4)$. The values of the Wald statistics are equal to 4.04 for the Markov switching model and 10.89 for the standard RBC. P-values associated with them are equal to 0.401 and 0.028 respectively. Therefore, I reject the hypothesis that the coefficients of the VAR(1) matrix A from the data on output and investment are equal to those implied by the standard RBC model at the 97% confidence level. At the same time I cannot reject the hypothesis that the coefficients of the matrix A from the data are equal to those implied by the Markov switching model even at the 60% confidence level. Hence, I conclude that the Markov switching model is a much better representation of the observed VAR(1) on output and investment than the standard RBC model.

The next step in comparing the two models is to check how well they match moments of the actual data. The way I calibrated the values of b, a and d (see Table 1) ensures that first moments are matched with the data in both models. Table 4 presents the standard deviations of the growth rates of per capita output, consumption and investment for the two models with the same quantities estimated from the actual data.

As can be seen from Table 4 both models do well in matching the variance of the growth rate of output. The RBC model does a little better in matching the variance of the growth rate of investment. The variance of the growth rate of consumption is matched only by the Markov switching model, it is well-known that the RBC model predicts consumption growth that is too smooth.

The last test of performance of the two models is to compare the cyclical characteristics of output produced by each of them with that in the data. Such cyclical characteristics could be the average length and amplitude of business cycle phases. For this purpose I use the dating algorithm developed in Harding and Pagan (1999). This algorithm first establishes the turning points of the cycle and then calculates average duration, average amplitude and average cumulative movements (that is the sum over the phase) of output contractions and expansions. Because this algorithm is designed to describe the "classical" (i.e. not detrended) cycle I add a trend of 2.1% per year to simulated output from both models before applying the dating algorithm. The number 2.1% was chosen because it is equal to the average growth rate of the sum of per capita private consumption and private investment (which is used as a measure of output in this paper) in quarterly US data for 1959.1 – 1999.2. Table 5 presents the results of this exercise.

It can be seen from Table 5 that both models perform similarly in characterizing the contraction phase of the cycle. They both predict shorter (by roughly 2 quarters) and less severe contractions than are in the data. However, they differ in their predictions about the expansions. Table 5 shows that the Markov switching model captures expansions relatively well (although it still underestimates the average amplitude), while the RBC model is unable to match this aspect of the data.

6. Conclusions

In this paper I constructed a dynamic rational expectations general equilibrium model of the business cycle which features a Markov switching utility function parameter as another source of uncertainty, thereby augmenting the supply side autoregressive technological shock of the classic real business cycle model with some “animal spirits” effects. I also compared the properties of this model with the real business cycle model by using US quarterly data for the period of 1959 – 1999. Several conclusions emerge.

First, the Markov switching model performs better than the standard RBC model in replicating the first order VAR for US quarterly data on output and investment. The Wald test allows me to reject the hypothesis that the VAR(1) representation of the data is the same as implied by the standard RBC model at the 97% confidence level. At the same time I am unable to reject the hypothesis that the VAR(1) representation of the data is the same as implied by the Markov switching model even at the 60% confidence level.

Second, both models can match the volatility of the growth rate of US output and investment. The Markov switching model also predicts a volatility of the consumption growth rate which is much closer to that of the US data than the volatility of consumption growth rate predicted by the RBC model, since the latter predicts too smooth consumption.

Third, both models perform similarly in characterizing output contractions. They both predict shorter and less severe contractions than in history. However, the Markov switching model captures the dynamics of expansions relatively well while the RBC model fails to do so.

Appendix. Derivation of coefficient a_1 –a_6 in equations (20) – (21)

Equation (18) can be used to express l_t as a function of c_t, k_t, and z_t. Using this expression one can write the marginal utility of consumption as the following:

$$
\begin{aligned}
&-r_t c_t + d_L l_t = -(r_t + d_L \frac{1-1/r_t}{s+a})c_t + d_L a \frac{1-1/r_t}{s+a} k_t + d_L (1-a)\frac{1-1/r_t}{s+a} z_t + \\
&d_L \frac{1-1/r_t}{s+a}\log\frac{r_t}{r^*} = A_t + B_t k_t + D_t z_t + d_L \frac{1-1/r_t}{s+a}\log\frac{r_t}{r^*},
\end{aligned} \tag{22}
$$

where A_t, B_t, D_t are some constants equal to A_0, B_0, C_0 in the state 0 and A_1, B_1, C_1 in the state 1. Substitution of (22) into (16) yields the following:

$$
\begin{aligned}
&A_t + B_t k_t + D_t z_t = \\
&E_t[A_{t+1} + m\log\frac{r_{t+1}}{r^*} + (B_{t+1} - ms)k_{t+1} + (C_{t+1} + m(1+s))z_{t+1} - mc_{t+1}] + M_t,
\end{aligned} \tag{23}
$$

where $m = \frac{q+d}{1+q}$ and $M_t = E_t[\frac{d_L}{s+a}\log\frac{r_{t+1}}{r_t} - m(r_{t+1} - r_t)]$. Equation (22) can be used to express consumption c_{t+1} as the following:

$$
c_{t+1} = -\frac{A_{t+1}}{x_{t+1}} - \frac{B_{t+1} - \frac{ad_L}{s+a}(1-\frac{1}{r_{t+1}})}{x_{t+1}}k_{t+1} - \frac{D_{t+1} - \frac{(1-a)d_L}{s+a}(1-\frac{1}{r_{t+1}})}{x_{t+1}}z_{t+1}, \tag{24}
$$

where $x_{t+1} = r_{t+1} + \frac{d_L}{s+a}(1-\frac{1}{r_{t+1}})$. Substitution of (24) into (23) gives the following:

$$
\begin{aligned}
&A_t + B_t k_t + D_t z_t = \\
&E_t[A_{t+1}(1+\frac{m}{x_{t+1}}) + m\log\frac{r_{t+1}}{r^*} + (B_{t+1}(1+\frac{m}{x_{t+1}}) - m(s + (1-\frac{1}{r_{t+1}})\frac{d_L a}{(s+a)x_{t+1}}))k_{t+1} + \\
&(D_{t+1}(1+\frac{m}{x_{t+1}}) + m(1+s-(1-\frac{1}{r_{t+1}})\frac{d_L(1-a)}{(s+a)x_{t+1}}))z_{t+1}] + M_t.
\end{aligned} \tag{25}
$$

Substitution of expression for c_t from (24) and expression for l_t from (18) into (17) gives the following representation of k_{t+1}:

$$k_{t+1}=dk_t+A_t\frac{ck}{x_t}+(ak+\frac{B_t-\frac{a}{s+a}d_L(1-\frac{1}{r_t})}{x_t}ck)k_t+(bk+\frac{D_t-\frac{1-a}{s+a}d_L(1-\frac{1}{r_t})}{x_t}ck)z_t, \quad (26)$$

where $ak=1-d+(1-\frac{1-a}{s+a}s)\frac{Y^*}{K^*}$, $bk=\frac{(1-a)(1+s)}{s+a}\frac{Y^*}{K^*}$, $ck=\frac{1+s}{s+a}\frac{Y^*}{K^*}-d$,

$dk_i=\frac{1-a}{s+a}bk\log\frac{r^i}{r^*}$, i=1,2. Substitution of (26) into (25) and using the fact that $E_t z_{t+1}=j\ z_t$ will allow me to represent both sides of equation (25) as functions of k_t and z_t. Equating coefficients before k_t on the left and right –hand sides of (25) gives the following system:

$$\begin{cases} B_0=(ak+\frac{B_0-\frac{ad_L}{s+a}(1-\frac{1}{r^0})}{x_0}ck)\times[p_0(B_0(1+\frac{m}{x_0})-m(s+(1-\frac{1}{r^0})\frac{ad_L}{(s+a)x_0}))+ \\ (1-p_0)(B_1(1+\frac{m}{x_1})-m(s+(1-\frac{1}{r^1})\frac{ad_L}{(s+a)x_1}))], \\ B_1=(ak+\frac{B_1-\frac{ad_L}{s+a}(1-\frac{1}{r^1})}{x_0}ck)\times[p_1(B_1(1+\frac{m}{x_1})-m(s+(1-\frac{1}{r^1})\frac{ad_L}{(s+a)x_1}))+ \\ (1-p_1)(B_0(1+\frac{m}{x_0})-m(s+(1-\frac{1}{r^0})\frac{ad_L}{(s+a)x_0}))].. \end{cases} \quad (27)$$

(27) can be reduced to one polynomial of fourth order and, therefore, has four roots. The saddle – path property of the model ensures that only one of these solutions is not explosive. After B_0 and B_1 are found A_0, A_1, C_0, C_1 can be found as solutions of the following linear systems:

$$\begin{bmatrix} p_0(1+\frac{m}{x_0})+Q_0\frac{ck}{x_0}-1 & (1-p_0)(1+\frac{m}{x_1}) \\ (1-p_1)(1+\frac{m}{x_0}) & p_1(1+\frac{m}{x_1})+Q_1\frac{ck}{x_1}-1 \end{bmatrix}\begin{bmatrix} A_0 \\ A_1 \end{bmatrix}=\begin{bmatrix} M^0+Q_0dk_0+m\log\frac{(r^0)^{p_0}(r^1)^{1-p_0}}{r^*} \\ M^1+Q_1dk_1+m\log\frac{(r^1)^{p_1}(r^1)^{1-p_1}}{r^*} \end{bmatrix},$$

$$\begin{bmatrix} j p_0 (1+\frac{m}{x_0}) + Q_0 \frac{ck}{x_0} - 1 & j(1-p_0)(1+\frac{m}{x_1}) \\ j(1-p_1)(1+\frac{m}{x_0}) & j p_1 (1+\frac{m}{x_1}) + Q_1 \frac{ck}{x_1} - 1 \end{bmatrix} \begin{bmatrix} D_0 \\ D_1 \end{bmatrix} =$$

$$\begin{bmatrix} Q_0 (bk - (1-\frac{1}{r_0})\frac{(1-a)d_L ck}{(s+a)x_0}) + j(m(1+s) + \frac{(1-a)d_L}{s+a}(\frac{p_0}{x_0} + \frac{1-p_0}{x_1})) \\ Q_1 (bk - (1-\frac{1}{r_1})\frac{(1-a)d_L ck}{(s+a)x_1}) + j(m(1+s) + \frac{(1-a)d_L}{s+a}(\frac{p_1}{x_1} + \frac{1-p_1}{x_0})) \end{bmatrix},$$

where

$$M^i = E[M_t \mid r_t = r^i],$$

$$Q_0 = p_0[B_0(1+\frac{m}{x_0}) - m(s+\frac{ad_L}{(s+a)x_0})(1-\frac{1}{r^0})] + (1-p_0)B_1(1+\frac{m}{x_1})m(s+\frac{ad_L}{(s+a)x_1})(1-\frac{1}{r^1}),$$

$$Q_1 = p_1[B_1(1+\frac{m}{x_1}) - m(s+\frac{ad_L}{(s+a)x_1})(1-\frac{1}{r^1})] + (1-p_1)B_1(1+\frac{m}{x_0})m(s+\frac{ad_L}{(s+a)x_0})(1-\frac{1}{r^0}).$$

Equation (24) can then be used to find the coefficients a_1 –a_6 in (20) – (21).

Note

[1]The Kroneker product of two matrices $A=\{a_{ij}\}$, $i=1\div n_1$, $j=1\div m_1$ and $B=\{b_{ij}\}$, $i=1\div n_2$, $j=1\div m_2$ is the $n_1n_2 x m_1m_2$ matrix given by $A \otimes B = \begin{bmatrix} a_{11}B \ldots\ldots a_{1m_1}B \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n_1 1}B \ldots\ldots a_{n_1 m_1}B \end{bmatrix}$.

Table 1. Calibrated parameters of the model

Parameter	Description	Value
q	Subjective discount rate	0.8%
b	Subjective discount factor	0.992
Y/K^*	Output – capital ratio	0.091
C/Y^*	Consumption – output ratio	0.80
a	Capital share	0.28
d	Capital depreciation rate	1.9%
j	Technological shock persistence	0.92
s	Inverse of labor supply elasticity	0

Table 2. Estimated parameters of alternative models.

Parameter	Description	Markov switching model	RBC model
r_0	Risk aversion parameter	1.00	1.29
r_1	Risk aversion parameter	1.08	1.29
p_0	Transitional probability	0.989	-------
p_1	Transitional probability	0.989	-------
s	Standard deviation of shock to technology process	0.49%	0.85%
m	Utility function constant	-0.78	-------

Table 3. VAR estimates for simulated and actual data.

Variable	US Data	Markov switching model	RBC model
a_{11}	0.99	0.98	0.90
a_{12}	-0.01	-0.02	0.01
a_{21}	0.16	0.06	-0.30
a_{22}	0.85	0.86	0.98
s_{11}	0.04	0.02	0.04
s_{12}	0.02	0.01	0.01
s_{21}	0.15	0.07	0.14
s_{22}	0.06	0.03	0.04
w_{11}	$1.30*10^{-4}$	$1.51*10^{-4}$	$1.36*10^{-4}$
w_{12}	$4.67*10^{-4}$	$4.62*10^{-4}$	$5.05*10^{-4}$
w_{22}	$22.45*10^{-4}$	$15.50*10^{-4}$	$18.77*10^{-4}$

Table 4. Selected standard deviations of actual and simulated data.

Variable	US Data	Markov switching model	RBC model
Output growth rate	1.26%	1.15%	1.19%
Consumption growth rate	0.67%	0.73%	0.37%
Investment growth rate	4.08%	4.87%	4.47%

Table 5. Actual and simulated business cycles characteristics of output.

	US Data	Markov switching model	RBC model
Mean duration (quarters)			
Peak to through	5.7	2.9	3.2
Trough to peak	14.3	14.4	10.3
Mean amplitude (%)			
Peak to through	-4.6	-1.9	-2.1
Trough to peak	13.8	9.8	8.4
Cumulation (%)			
Peak to through	-12.9	-4.6	-4.9
Trough to peak	139	117	62.6

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